WEST BENGAL STATE UNIVERSITY

B.Sc. Honours/Programme 3rd Semester Supplementary Examination, 2021

MTMHGEC03T/MTMGCOR03T-MATHEMATICS (GE3/DSC3)

Time Allotted: 2 Hours | Full Marks: 50

The figures in the margin indicate full marks.
Candidates should answer in their own words and adhere to the word limit as practicable.
All symbols are of usual significance.

Answer Question No. 1 and any *five* from the rest

1. Answer any ***five*** questions from the following: 2×5 = 10

(a) Prove that the set $\mathbb{Z}$ (set of integers) is countable.

(b) Show that every real number is a cluster point of the set of rational numbers.

(c) Give an example of divergent sequences $\{u_n\}$ and $\{v_n\}$ such that sequence $\{u_n + v_n\}$ is convergent.

(d) Show that every bounded sequence is not convergent.

(e) Test the convergence of the series $\sum_{n=1}^{\infty} u_n$, where $u_n = \frac{n}{2^n}$.

(f) Prove that for a fixed value of x the series, $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ is absolutely convergent.

(g) Use Weierstrass' M-test to show that the series $\sum_{n=1}^{\infty} \frac{n^5+1}{n^7+3}\left(\frac{x}{2}\right)^n$ converges uniformly in [2, 2].

(h) Determine the radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{n^n x^n}{n!}$.

2. (a) Prove that the interval (0, 1) is uncountable. 5

(b) Show that $\mathbb{N}$ is unbounded above. 3

3. (a) State and prove the Archimedean property of $\mathbb{R}$. Hence prove that, if $x \in \mathbb{R}$ then there exists a natural number 'n' such that $n > x$. 4+2

(b) If 'y' be a positive real number show that there exists a natural number 'm' such that $0 < \frac{1}{2^m} < y$. 2

4. (a) Prove that a convergent sequence is bounded. Is the converse true? — Justify. 3+1

(b) Prove that if the sequence $\{u_n\}$ be bounded and $\lim_{n\to\infty} v_n = 0$, then $\lim_{n\to\infty} u_n v_n = 0$. 4

5. (a) State and prove the Cauchy's first theorem on limit. 4

(b) Prove that the sequence $\left\{\left(1+\frac{1}{n}\right)^{n+1}\right\}$ is a monotone decreasing sequence and find its limit. 4

6. (a) Prove that, the series $\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges for $p>1$ and diverges for $p\leq 1$. 4

(b) Examine the convergence of the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2n+1}{n(n+1)}$. 4

7. (a) Define absolute convergence and conditional convergence of a series. Explore with an example. 3+1

(b) Examine the convergence of the following series 2+2

(i) $\frac{3}{2}+\frac{4}{2^2}+\frac{5}{2^3}+\cdots\cdots+\frac{n+2}{2^n}+\cdots\cdots$

(ii) $1-\frac{1}{2!}+\frac{1}{3!}-\frac{1}{4!}+\cdots\cdots+\frac{(-1)^{n+1}}{n!}+\cdots\cdots$.

8. (a) Let $\{f_n(x)\}$ be a sequence of functions on $D(\subset\mathbb{R})$ and pointwise convergent on D to a function $f(x)$. Let $M_n=\sup_{x\in D}|f_n(x)-f(x)|$. Then prove that $\{f_n(x)\}$ is uniformly convergent on D to $f(x)$ iff $\lim_{n\to\infty}M_n=0$. 4

(b) Let $f_n(x)=\frac{nx}{1+n^2x^2}$, $x\in[0, 1]$. Show that the sequence $\{f_n(x)\}$ does not converge uniformly on [0, 1]. 4

9. (a) Use Cauchy's general principle of convergence to show that the sequence $\left\{\frac{n}{n+1}\right\}$ is convergent. 4

(b) Assuming the power series expansion for $(1+x)^{-1}=1-x+x^2-x^3+\cdots\cdots$, $|x|<1$, show that $\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots\cdots$, $-1<x<1$. 4

N.B. : *Students have to complete submission of their Answer Scripts through E-mail / Whatsapp to their own respective colleges on the same day / date of examination within 1 hour after end of exam. University / College authorities will not be held responsible for wrong submission (at in proper address). Students are strongly advised not to submit multiple copies of the same answer script.*

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